ON THE STABILITY OF STEADY COMBUSTION OF SOLID FUELS

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In his paper [1] Zel'dovich had derived the dependence of the rate of steady combustion of a solid fuel on the initial temperature. Istratov and Librovich [2] had investigated the stability of steady combustion of solid fuels taking into consideration the variation of temperature at the surface of such fuels.

The effects of heat release in the reaction zones and of the variation of fields of temperature and of the rate [of combustion] on the stability of the steady state combustion of solid fuels are investigated in this paper. The assumptions as to the mechanism of solid fuel combustion on which this investigation is based are given in [2].

The method of small perturbations is used for analyzing the stability of steady combustion of solid fuels, and a new stability criterion is derived. Calculations carried out on a type BESM-2M computer had shown that stability is materially affected by the following parameters: σ, z_2/c_p, Θ_r, Δ_2/c_p, and $\nu\xi_2^\circ$.

The problem is solved in a univariate formulation. We assume the system of coordinates to be permanently attached to the boundary separating the solid fuel from its products of combustion. The x-axis is directed from the solid fuel face toward the burning surface.

Region 1 is defined by the heat conduction equation

$$\frac{\partial T_1}{\partial t} + u_1 \frac{\partial T_1}{\partial x} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \qquad a_1 = \frac{\lambda_1}{c_{p1}\rho_1}. \tag{1}$$

The processes taking place in region 2 are defined by the system of equations derived in [3]

$$\begin{gathered} \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = -\frac{1}{\rho_2}\frac{\partial p_2^*}{\partial x} + \frac{4}{3}\frac{1}{\rho_2}\frac{\partial}{\partial x}\left(\mu_2 \frac{\partial u_2}{\partial x}\right) \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial(\rho_2 u_2)}{\partial x} = 0, \qquad c_{p2}\left(\frac{\partial T_2}{\partial t} + u_2\frac{\partial T_2}{\partial x}\right) = \frac{1}{\rho_2}\frac{\partial}{\partial x}\left(\lambda_2 \frac{\partial T_2}{\partial x}\right) \\ \frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = \frac{1}{\rho_2}\frac{\partial}{\partial x}\left(D\rho_2 \frac{\partial s_2}{\partial x}\right). \end{gathered} \tag{2}$$

Region 3 containing gaseous products of combustion is defined by the equation of heat conduction

$$\frac{\partial T_3}{\partial t} + u_3 \frac{\partial T_3}{\partial x} = a_3 \frac{\partial^2 T_3}{\partial x^2}, \qquad a_3 = \frac{\lambda_3}{c_{p3}\rho_3}. \tag{3}$$

Here T, u, ρ, p^*, and s are, respectively, the temperature, the rate [of combustion], the pressure and the concentration; c_p, μ, D, and λ are, respectively, the specific heat, the dynamic viscosity coefficient, the diffusion coefficient, and the coefficient of thermal conductivity; and t is the time. Subscript 1 relates to the k-phase of the solid fuel, and subscripts 2 and 3 to the gaseous phases in regions 2 and 3, respectively.

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Let us write Eqs. (1)-(3) in a dimensionless form, introducing dimensionless parameters of the form

$$\theta = \frac{T - T_0}{T_b - T_0}, \qquad \xi = \frac{u_1^\circ \rho_1 c_{p1}}{\lambda_1} x, \qquad \tau = \frac{u_1^{\circ 2} \rho_1 c_{p1}}{\lambda_1} t,$$

$$R_2 = \frac{\rho_2}{\rho_3^\circ}, \qquad p = \frac{p^*}{\rho_3^\circ u_n^2}, \qquad U = \frac{u}{u_n}.$$

Here u_1° is the linear rate of steady combustion, u_n is the normal rate of steady flame propagation, ρ_3° is the density of gas in region 3 at steady combustion, T_0 is the initial temperature of the solid fuel, and T_b the temperature of combustion.

The system of equations (1)-(3) must be supplemented by the equation of state which in the case of an incompressible gas (the rate of flame [propagation] is small in comparison with the speed of sound) reduces to the relationship between the density and temperature of gas [4]

$$R_2 = \frac{1}{1 + \beta\theta_2}, \qquad \beta = \frac{1 - \alpha}{\alpha}. \tag{4}$$

Here α is the ratio of fuel and gas densities.

Introducing the mass rate of combustion m = RU and taking into consideration the equation of state (4), we can write the system of Eqs. (1)-(3) in the form

$$\begin{gathered}
\frac{\partial \theta_1}{\partial \tau} + m_1 \frac{\partial \theta_1}{\partial \xi} = \frac{\partial^2 \theta_1}{\partial \xi^2} \qquad \left(m_1 = \frac{\rho_1 u_1}{\rho_1 u_1^\circ}, \ \rho_1 u_1 = \rho_3^\circ u_n\right) \\
\varepsilon \frac{\partial m_2}{\partial \tau} + \varepsilon \frac{\beta m_2}{1 + \beta\theta_2} \frac{\partial \theta_2}{\partial \tau} + m_2 (1 + \beta\theta_2) \frac{\partial m_2}{\partial \xi} + \beta m_2^2 \frac{\partial \theta_2}{\partial \xi} = - \frac{\partial p_2}{\partial \xi} \\
+ \frac{4}{3} \frac{P}{\nu} \left[(1 + \beta\theta_2) \frac{\partial^2 m_2}{\partial \xi^2} + 2\beta \frac{\partial m_2}{\partial \xi} \frac{\partial \theta_2}{\partial \xi} + \beta m_2 \frac{\partial^2 \theta_2}{\partial \xi^2} \right] \\
- \frac{\varepsilon\beta}{(1 + \beta\theta_2)^2} \frac{\partial \theta_2}{\partial \tau} + \frac{\partial m_2}{\partial \xi} = 0, \qquad \varepsilon \frac{1}{1 + \beta\theta_2} \frac{\partial \theta_2}{\partial \tau} + m_2 \frac{\partial \theta_2}{\partial \xi} = \frac{1}{\nu} \frac{\partial^2 \theta_2}{\partial \xi^2} \\
\varepsilon \frac{1}{1 + \beta\theta_2} \frac{\partial s_2}{\partial \tau} + m_2 \frac{\partial s_2}{\partial \xi} = \frac{1}{\nu L} \frac{\partial^2 s_2}{\partial \xi^2}, \qquad \varepsilon \frac{\partial \theta_3}{\partial \tau} + \frac{\partial \theta_3}{\partial \xi} = \frac{1}{\nu} \frac{\partial^2 \theta_3}{\partial \xi^2} \\
\left(\varepsilon = \frac{\rho_3^\circ}{\rho_1}, \quad \lambda = \frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_1}, \quad c_p = \frac{c_{p2}}{c_{p1}} = \frac{c_{p3}}{c_{p1}}, \ \nu = \frac{c_p}{\lambda}\right).
\end{gathered} \tag{5}$$

Here P and L are, respectively, the Prandtl and the Lewis numbers.

Let us formulate boundary conditions. At infinity the solutions of Eqs. (5) must satisfy

$$\theta_1 \to 0 \quad \text{for } \xi \to -\infty, \qquad \theta_3 < \infty \quad \text{for } \xi \to +\infty \tag{6}$$

and, also, the conditions at the interfaces of regions. Let us assume that the interfaces of reactions may be subject to small displacements. In a stationary system of coordinates the coordinate of the reaction region in the solid fuel k-phase is

$$x_{n1} = x_{n0} - \int_0^t u_1 dt$$

where x_{n0} is the initial position of the solid fuel face. From this we have

$$u_1 = - dx_{n1} / dt.$$

Let us assume that

$$\xi_{n1} = \xi_{n0} - (\tau + f e^{\omega\tau}).$$

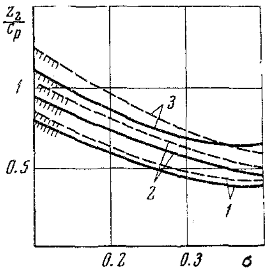

Fig. 1

Then

$$m_1 = 1 + \omega f e^{\omega\tau}.$$

In the moving system of coordinates $\xi_1 = 0$. We assume the coordinate of the reaction zone in the gas to be

$$\xi_2 = \xi_2^\circ + g e^{\tau\omega}.$$

Here ξ_2° is the [coordinate of the] steady position of the reaction zone. According to [3] the width of region 2 is of the order $\mu_2/\rho_2 u_2$. Hence

$$\xi_2^\circ \sim \frac{u_1^\circ \rho_1 c_{p1}}{\lambda_1} \frac{\mu_2}{\rho_2 u_2} = \frac{\rho_1 u_1^\circ}{\rho_2 u_2} \frac{P}{\nu} \qquad \left(P = \frac{\mu_2 c_{p2}}{\lambda_2}\right).$$

For a steady combustion process $\rho_1 u_1^\circ = \rho_2 u_2$. We then have

$$\xi_2^\circ \sim P/\nu. \tag{7}$$

Let us assume, as in [4], that variations of the chemical reaction rates w_1 and w_2 resulting from a perturbation of the steady state are dependent on variations of temperatures T_s and T_r at the corresponding faces of the k-phase and of the gas. This is a reasonable assumption in the case of high energies E_1 and E_2 of chemical reaction activation. The equations for reaction rates in the k-phase and in the gas may now be written as

$$m_1 = \frac{\rho_1 w_1}{\rho_1 u_1^\circ} = 1 + z_1(\theta_s^\circ + \theta_s'), \qquad z_1 = \frac{E_1(T_b - T_0)}{2RT_s^2}$$
$$m_2 = \frac{\rho_2 w_2}{\rho_1 u_1^\circ} = 1 + z_2(\theta_r^\circ + \theta_r'), \qquad z_2 = \frac{E_2(T_b - T_0)}{2RT_b^2}.$$

The boundary conditions at interface of regions, expressed in dimensionless form, are of the form

$$\begin{aligned}
&\text{for } \xi_1 = 0 \\
&\theta_1 = \theta_2 = \theta_s, \quad \frac{\partial\theta_1}{\partial\xi} - \lambda\frac{\partial\theta_2}{\partial\xi} - (\theta_s + \theta_0^\circ)(1 - c_p)m_1 = \frac{q_1 m_1}{c_{p1}(T_b - T_0)}, \\
&m_1 = m_2 = 1 + \omega f e^{\omega\tau}, \quad m_2 s_2 - \frac{1}{\nu L}\frac{\partial s_2}{\partial\xi} = m_1 \quad \left(\theta_0^\circ = \frac{T_0}{T_b - T_0}\right); \\
&\text{for } \xi = \xi_2 \\
&\theta_2 + \xi_2\frac{\partial\theta_2}{\partial\xi} = \theta_3 = \theta_r, \quad \lambda\frac{\partial}{\partial\xi}\left(\theta_2 + \xi_2\frac{\partial\theta_2}{\partial\xi}\right) - \lambda\frac{\partial\theta_3}{\partial\xi} = \frac{q_2 m_1}{c_{p1}(T_b - T_0)}, \\
&s_2 + \xi_2\frac{\partial s_2}{\partial\xi} = 0, \quad -\frac{1}{\nu L}\left(\frac{\partial s_2}{\partial\xi} + \xi_2\frac{\partial^2 s_2}{\partial\xi^2}\right) = m_2.
\end{aligned} \tag{8}$$

Here q_1 and q_2 are the reaction heat effects in the k-phase and in gas, respectively.

We thus have a system of six differential equations (5) in six unknowns θ_1, θ_2, θ_3, p_2, m_2, and s_2, whose solution must satisfy boundary conditions (6) and (8).

Since in the following the value of p_2 will not be required, the second of Eqs. (5) may be omitted from further considerations.

We use the method of small perturbations for analyzing stability, and shall seek the solution of the derived system of equations in the form

$$f(\xi, \tau) = F(\xi) + [\varepsilon^\circ f^\circ(\xi) + \varepsilon f'(\xi)]\, e^{\xi}. \tag{9}$$

Substituting the solution of the form (9) into Eqs. (5) and into boundary conditions (6) and (8), retaining in these terms of the form $F(\xi)$ and equating coefficients of ε°, we obtain a system of equations and boundary conditions defining the steady process and the perturbed state. Solution of the steady [state] problem is of the form

$$\theta_1 = \theta_s e^{\xi}, \; M_2 = \text{const} = 1$$

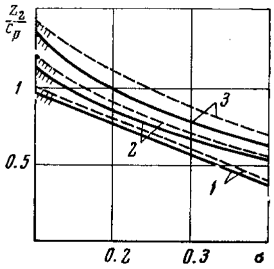

Fig. 2

$$\Theta_2 = \Theta_r - \frac{\Delta_2}{c_p}\left[1 - \frac{1}{1+\nu\xi_2^\circ}\exp\nu(\xi-\xi_2^\circ)\right] \quad \left(\Delta_i = \frac{q_i}{c_{p1}(T_b - T_0)},\quad i = 1, 2\right)$$

$$S_2 = 1 - \frac{1}{1+\nu L\xi_2^\circ}\exp\nu L(\xi - \xi_2^\circ), \qquad \Theta_3 = \Theta_r \tag{10}$$

$$\left(\Theta_s = \frac{1}{c_p}\left[\Delta_1 + \Delta_2\frac{e^{\nu\xi_2^\circ}}{1+\nu\xi_2^\circ} + \theta_0^\circ(1-c_p)\right], \qquad \Theta_2 = \frac{1}{c_p}\left[\Delta_1 + \Delta_2 + \theta_0^\circ(1-c_p)\right]\right).$$

After reduction of the common factor $e^{\omega\tau}$, the system of equations defining the perturbed state can be written as

$$\frac{d^2\theta_1^\circ}{d\xi^2} - \frac{d\theta_1^\circ}{d\xi} - \omega\theta_1^\circ = -\omega f\Theta_s e^{\xi}, \qquad \frac{1}{\nu}\frac{d^2\theta_2^\circ}{d\xi^2} - \frac{d\theta_2^\circ}{d\xi} = m_2^\circ\frac{d\Theta_2}{d\xi} \tag{11}$$

$$\frac{dm_2^\circ}{d\xi} = 0, \qquad \frac{1}{\nu L}\frac{d^2 s_2^\circ}{d\xi^2} - \frac{ds_2^\circ}{d\xi} = m_2^\circ\frac{dS_2}{d\xi}, \qquad \frac{1}{\nu}\frac{d^2\theta_3^\circ}{d\xi^2} - \frac{d\theta_3^\circ}{d\xi} = 0\,. \tag{12}$$

The boundary conditions are

$$\theta_1^\circ \to 0 \quad \text{for } \xi \to -\infty, \qquad \theta_3^\circ < \infty \quad \text{for } \xi \to +\infty;$$

for $\xi = 0$

$$\theta_1^\circ = \theta_2^\circ = \theta_s^\circ, \quad \frac{d\theta_1^\circ}{d\xi} - \lambda\frac{d\theta_2^\circ}{d\xi} - \theta_s^\circ(1-c_p) - \omega f(\Theta_s + \theta_0^\circ)(1-c_p) = \Delta_1 z_1\theta_s^\circ,$$

$$m_1^\circ = m_2^\circ = \omega f, \qquad z_2\theta_s^\circ = \omega f, \qquad s_2^\circ + \omega f S_2 - \frac{1}{\nu L}\frac{ds_2^\circ}{d\xi} = z_1\theta_s^\circ;$$

for $\xi = \xi_2$

$$\theta_2^\circ + g\frac{d\Theta_2}{d\xi} + \xi_2^\circ\frac{d\theta_2^\circ}{d\xi} = \theta_3^\circ = \theta_r^\circ, \quad \lambda\frac{d\theta_2^\circ}{d\xi} + \lambda g\frac{d^2\Theta_2}{d\xi^2} + \lambda\xi_2^\circ\frac{d^2\theta_2^\circ}{d\xi^2} - \lambda\frac{d\theta_3^\circ}{d\xi} = \Delta_2 z_2\theta_r^\circ,$$

$$s_2^\circ + g\frac{dS_2}{d\xi} + \xi_2^\circ\frac{ds_2^\circ}{d\xi} = 0, \qquad -\frac{1}{\nu L}\left[\frac{ds_2^\circ}{d\xi} + g\frac{d^2S_2}{d\xi^2} + \xi_2^\circ\frac{d^2s_2^\circ}{d\xi^2}\right] = z_2\theta_r^\circ.$$

The solution of system (11) with conditions (12) is

$$\theta_1^\circ = ae^{r\xi} + f\Theta_s e^{\xi}, \qquad r = {}^1/_2(1 + \sqrt{1+4\omega}), \qquad m_0^2 = \text{const} = \omega f$$

$$\theta_2^\circ = b + c\exp\nu\xi + f\frac{\Delta_2}{\lambda}\frac{\omega\xi}{1+\nu\xi_2^\circ}\exp\nu(\xi - \xi_2^\circ) \tag{13}$$

$$s_2^\circ = k + l\exp\nu L\xi - f\frac{\nu L\omega\xi}{1+\nu L\xi_2^\circ}\exp\nu L(\xi - \xi_2^\circ).$$

Substituting solutions (13) into boundary conditions (12), we obtain a system of homogeneous equations for the determination of constants, a, b, c, d, f, g, k, and l. For a nontrivial solution to exist the determinant of this system must be equal to zero. The equation defining the zero approximation of the complex frequency is of the form

$$2\omega(y-1) + \sigma\omega + y = (y + \sigma\omega)\sqrt{1+4\omega} \tag{14}$$

where

$$y = \frac{z_2}{c_p}\Theta_s, \qquad \sigma = \frac{z_2}{z_1 c_p}\,. \tag{15}$$

After simple transformations, the frequency equation may be written as

$$\omega^3 + \eta_2\omega^2 + \eta_1\omega = 0 \tag{16}$$

where

$$\eta_1 = \sigma^{-2}y, \quad \eta_2 = \sigma^{-2}[-(y-1)^2 + \sigma(y+1)]\,.$$

The roots of this equation are

$$\omega_1 = 0, \qquad \omega_{2,3} = -{}^1/_2\,\eta_2 \pm {}^1/_2\sqrt{\eta_2^2 - 4\eta_1}\,. \tag{17}$$

Since the solution of Eq. (14) had necessitated a squaring operation, its extraneous roots must be eliminated by substitution of obtained roots into the input equation.

The perturbations corresponding to the root $\omega_1 = 0$ occur when the initial steady state distribution of parameters shifts along the x-axis without altering its form. According to [5] the determination of stability must be invariant with respect to this transformation owing to the invariance of the problem formulation with respect to shift along the x axis.

When $(\eta_2^2 - 4\eta_1) < 0$, the roots of Eq. (16) are complex. It follows from (17) that in this case the solution defines a steady process ($\text{Re}\,\omega < 0$), when

$$[-(y-1)^2 + \sigma(y+1)] > 0. \tag{18}$$

From (10) follows

$$\Theta_s = \Theta_r - \frac{\Delta_2}{c_p}\left(1 - \frac{\exp \nu\xi_2^\circ}{1 + \nu\xi_2^\circ}\right). \tag{19}$$

Thus, under conditions of steady combustion, the temperature at the solid fuel face depends on Θ_r, Δ_2/c_p, and $\nu\xi_2^\circ$, i.e., on the properties of the gaseous phase.

Substituting relationship (19) into (15), we obtain

$$y = \frac{z_2}{c_p}\left[\Theta_r - \frac{\Delta_2}{c_p}\left(1 - \frac{1}{1 + \nu\xi_2^\circ}\exp \nu\xi_2^\circ\right)\right] \qquad (\nu\xi_2^\circ \sim P).$$

Condition (18) derived for the stability of solid fuel combustion depends on Θ_r, z_2/c_p, σ, Δ_2/c_p, and $\nu\xi_2^\circ$. Unlike in the stability criteria formulated earlier [1, 2], Θ_s is determined here by solving the steady-state problem.

A BESM-2M computer was used for estimating the effect of these parameters on the stability of solid fuel combustion in the following range of variation of basic parameters:

$$0.1 \leqslant \sigma \leqslant 0.4,\ \ 0 \leqslant z_2/c_p \leqslant 2.5,\ \ 0.8 \leqslant \nu\xi_2^\circ \leqslant 2.5,\ \ 0.3 \leqslant \Delta_2/c_p \leqslant 0.85,\ \ \Theta_r = 1.$$

Results of calculations are shown in Figs. 1 and 2. Curves 1, 2, and 3 in Fig. 1 relate to $\Delta_2/c_p = 0.3$, 0.5, and 0.6, where the solid and the dashed lines correspond, respectively, to $\nu\xi_2^\circ = 0.8$, and 1.1. In Fig. 2 curves 1, 2, and 3 relate to $\Delta_2/c_p = 0.4$, 0.5, and 0.6 with the solid and dashed lines corresponding, respectively, to $\nu\xi_2^\circ = 1.5$ and 2.5. Regions of instability are indicated by shading. These curves show that with increasing $\nu\xi_2^\circ$ at $\Delta_2/c_p = \text{const}$, as well as with increasing Δ_2/c_p at $\nu\xi_2^\circ = \text{const}$, the instability region shifts upward along the z_2/c_p-axis. Hence a widening of the preheat region (the dark zone) and increasing the reaction heat effect in the gas results in the shift of the stability region upwards in the direction of the z_2/c_p axis.

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